LAIKIPIA UNIVERSITY

UNIVERSITY EXAMINATIONS

SECOND SEMESTER 2023/2024 ACADEMIC YEAR

FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (GENERAL)

MATH 423: PARTIAL DIFFERENTIAL EQUATIONS II

STREAM: R ***TIME: 2 HRS***

DAY: THURSDAY [11.30A.M – 1.30P.M] ***DATE: 11/04/2024***

THIS QUESTION PAPER CONSISTS OF FOUR (4) PAGES

PLEASE DO NOT OPEN UNTIL THE INVIGILATOR SAYS SO.

INSTRUCTIONS:

1. **(i) Answer question ONE and any other TWO QUESTIONS in this paper, and NOT MORE.**
2. **(ii) Show all your working clearly**
3. **(iii) Do not write on the question paper**

QUESTION ONE (30 MARKS - COMPULSORY)

(a) Define a second order partial differential equation (PDE) with constant coefficients, and list at least two areas where differential equations are applied in real life. **(3 Marks)**

(b) By differentiating appropriately, obtain a first order partial differential equation associated with the relation $z = \alpha x^4 + \beta y^4$ **(4 Marks)**

(c) Solve the following basic PDEs

(i) $r = 7x$ **(3 Marks)**

(ii) $t = -\cos(6xy)$ **(5 Marks)**

(d) Classify the following PDEs in terms of order and degree

i. $\frac{\partial^2 z}{\partial x^2} - xy\frac{\partial^2 z}{\partial x \partial y} + 2 = x^2 y$ **(2 Marks)**

ii. $\sqrt{5 - \frac{\partial^6 y}{\partial x^6} =} \quad y^5$ **(3 Marks)**

(e) Solve **completely** the following non homogeneous PDE : $(D_1^2 - D_2^2)z = x^2 - 2y$ **(10 Marks)**

QUESTION TWO (20 MARKS - OPTIONAL)

a) Solve the PDE $\frac{\partial^2}{\partial x^2} z = 12x^2(t+1)$ given the boundary conditions that when $x = 0$ $, z = \cos 2t$ and $\frac{\partial z}{\partial x} = \sin t$ (HINT: use direct integration) **(8 Marks)**

b) A stretched string of length 20cm is set to oscillate by displacing its midpoint 1cm from its rest position and releasing it with zero initial velocity. By solving the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, $c^2=1$, determine that the resulting motion takes the form $u(x.t) = \sum_{r=1}^{\infty} \sin\frac{r\pi x}{20}\left[P_r \cos\frac{r\pi t}{20} + Q_r \cos\frac{r\pi t}{20}\right]$ **(12 Marks)**

QUESTION THREE (20 MARKS - OPTIONAL)

(a) A consequence of the relation, $z = f(u) + g(v) + h(x, y)$, with $u = u(x, y)$ *and* $v = v(x, y)$ is the matrix

$$\begin{vmatrix} u_x & v_x & 0 & 0 & h_x - p \\ u_y & v_y & 0 & 0 & h_y - q \\ u_{xx} & v_{xx} & u_x^{\ 2} & v_x^{\ 2} & h_{xx} - r \\ u_{xy} & v_{xy} & u_x u_y & v_x v_y & h_{xy} - s \\ u_{yy} & v_{yy} & u_y^{\ 2} & v_y^{\ 2} & h_{yy} - t \end{vmatrix} = 0,$$

which when solved, yields a second order PDE associated with that relationship. Using these facts, form a second order PDE from the relation $z = f(u) + g(v)$, where $u = 2x + y$ and $v = 5x + y$. **(10 Marks)**

(b) (i) Prove that when a non-homogeneous second order partial differential equation takes the form $F(D_1, D_2)z = cos(ax + by)$ or $F(D_1, D_2)z = sin(ax + by)$, then

$D_1^{\ 2} = -(a^2), \quad D_2^{\ 2} = -(b^2) \ \ and \ \ D_1 D_2 = -(ab)$ **(4 Marks)**

(ii) Use this result to solve **completely** the PDE $(D_1 + 2D_2 + 3)(D_1 - D_2)z = sin(2x - y)$

(6 Marks)

QUESTION FOUR (20 MARKS - OPTIONAL)

(a) You are required to use canonical transformation method to solve the PDE $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$

i) Find the discriminant and classify this equation **(3 Marks)**

ii) Determine the auxiliary equation and solve it to obtain the characteristic roots

(4 Marks)

iii) Hence solve the equation **(5 Marks)**

(b) Define reducible and non-reducible PDEs and classify each of the following equations accordingly

(i) $F(D_1, D_2) = D_1^{\ 2} + 4D_1 D_2 - 6D_2^{\ 2}$

(ii) $F(D_1, D_2) = D_1^{\ 2} + 3D_2^{\ 2}$

(iii) $F(D_1, D_2) = D_1^{\ 2} - D_2^{\ 2}$

(iv) $F(D_1, D_2) = D_1^{\ 2} - D_1 D_2 - 6D_2^{\ 2}$ **(8 Marks)**

QUESTION FIVE (20 MARKS - OPTIONAL)

a) Prove that for a reducible PDE of the form $F(D_1, D_2)z = (a_1 D_1 + b_1 D_2 + c_1)z = 0$, the complementary solution becomes $z = e^{-\frac{c_1}{a_1}x}(b_i x - a_i y)\phi_1$ **(6 Marks)**

(b) Further, if the PDE is non-homogeneous and takes the exponential form $F(D_1, D_2)z = e^{(ax+by)}$, prove that the consequence is that $D_1 = a$ and $D_2 = b$. **(4 Marks)**

(c) Combining the results of (a) and (b) above, obtain the **complete solution** of the following non-homogeneous equation $(pr - 2rq - pt + 2qt) = e^{(x+2y)}$ i.e. $(D_1^3 - 2D_1^2 D_2 - D_1\ D_2^2 + 2D_2^3)z = e^{(x+y)}$. (HINT: One of the factors of $F(D_1, D_2)$ is $(D_1 - 2D_2)$. By using ordinary long division, you should easily get the other factors). **(10 Marks)**